The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Final Exam ***Solutions 2.0***

Date: December 12, 2024 Course: ECE 313 Evans

Name: ____
Last, First

- **Exam duration**. The exam is scheduled to last two hours.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **No AI tools allowed**. As mentioned on the course syllabus, you may not use GPT or other AI tools during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

Problem	Point Value	Your Score	Topic
1	18		Continuous-Time System Properties
2	18		Discrete-Time Convolution
3	16		Continuous-Time Filter Design
4	18		Discrete-Time Filter Design
5	16		Continuous-Time Sinusoidal Amplitude Modulation
6	14		Discrete-Time Mystery Systems
Total	100		

Problem 1. Continuous-Time System Properties. *18 points*

Each continuous-time system has input $x(t)$ and output $y(t)$, and $x(t)$ and $y(t)$ might be complex-valued.

Determine if each system is linear or nonlinear, time-invariant or time-varying, and bounded-input bounded-output (BIBO) stable or unstable.

You must either prove that the system property holds in the case of linearity, time-invariance, or stability, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Part	*System Name*	*System Formula*	*Linear?*	*Time-Invariant?*	*BIBO Stable?*
(a)	Gain	$y(t) = A\,x(t)$ where A is finite constant. for $-\infty < t < \infty$	**Yes**	**Yes**	**Yes**
(b)	Tangent	$y(t) = \tan(\,x(t)\,)$ for $-\infty < t < \infty$	**No**	**Yes**	**No**
(c)	Scale time axis	$y(t) = x(2\,t)$ for $-\infty < t < \infty$	**Yes**	**No**	**Yes**

***Linearity*. We'll first apply the all-zero input test. If the output is not zero for all time, then the system is not linear. Otherwise, we'll have to apply the definitions for homogeneity and additivity. All-zero input test is a special case of homogeneity $a\,x(t) \to a\,y(t)$ when the constant $a = 0$.**

***Time-Invariance*: If the current output value $y(t)$ depends only on current input $x(t)$ and not on any other input/output values, it is pointwise operation. Pointwise operations are time-invariant.**

***BIBO Stability*. Bounded input $|\,x(t)\,| \le B < \infty$ would give bounded output $|\,y(t)\,| \le C < \infty$.**

(a) Gain: $y(t) = a\,x(t)$ for $-\infty < t < \infty$. Here, a is a finite constant. *6 points*.

***Linearity*: Passes all-zero input test. Need to check the following properties:**

- ***Homogeneity:* $y_{scaled}(t) = A\left(a\,x(t)\right) = a\left(A\,x(t)\right) = a\,y(t)$. YES.**
- ***Additivity*: $y_{additive}(t) = A\,(x_1(t) + x_2(t)) = A\,x_1(t) + A\,x_2(t) = y_1(t) + y_2(t)$. YES.**

***Time-Invariance*: All pointwise operations are time-invariant. See above. YES.**

***BIBO Stability*. $|\,y(t)\,| = |\,a\,x(t)\,| = |a|\,|\,x(t)\,| \le |a|\,B < \infty$. YES.**

(b) Tangent: $y(t) = \tan(\,x(t)\,)$ for $-\infty < t < \infty$. *6 points*.

***Linearity*: Passes all-zero input test. Need to check the following properties:**

- ***Homogeneity:* Let $x(t) = \frac{\pi}{4}$ so $y(t) = \tan\frac{\pi}{4} = 1$. When $x(t) = \frac{\pi}{2}$, $y(t) = \tan\frac{\pi}{2} = \infty$. NO.**

***Time-Invariance*: All pointwise operations are time-invariant. See above. YES.**

***BIBO Stability*. When $x(t) = \frac{\pi}{2}$, $y(t) = \tan\frac{\pi}{2} = \infty$. NO.**

(c) Scale time axis: $y(t) = x(2\,t)$ for $-\infty < t < \infty$. *6 points*. **CT version of F24 Midterm 2.1(c).**

***Linearity*: Passes the all-zero input test.**

***Homogeneity:* $y_{scaled}(t) = \left(a\,x(t)\right)_{t\to 2t} = a\,x(2t) = a\,y(t)$. YES.**

***Additivity*: $y_{additive}(t) = (x_1(t) + x_2(t))_{t\to 2t} = x_1(2t) + x_2(2t) = y_1(t) + y_2(t)$. YES.**

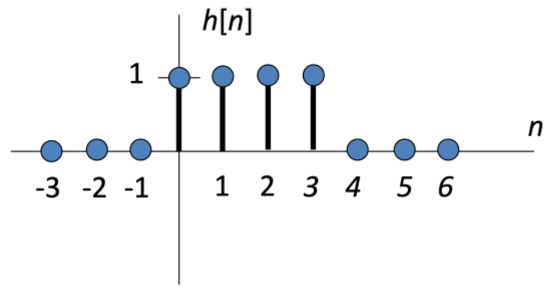
***Time-Invariance*: Let $t = n$. $y(t) = x(2t)$ selects even-indexed values of $x[n]$: $\{\ \ldots, x[-2], x[0], x[2], \ldots.\}$.**

Input $x[n-1]$. Output $y_{shifted}[n]$ will be $\{\ \ldots, x[-3], x[-1], x[1], \ldots.\}$. This is not $y[n-1]$. NO.

***BIBO Stability*. $|\,y[n]\,| = |\,x[2\,n]\,| \le B < \infty$. YES**

Problem 2. Discrete-Time Convolution. *18 points*

Consider a discrete-time linear time-invariant (LTI) system with impulse response plotted on the right of $h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$.

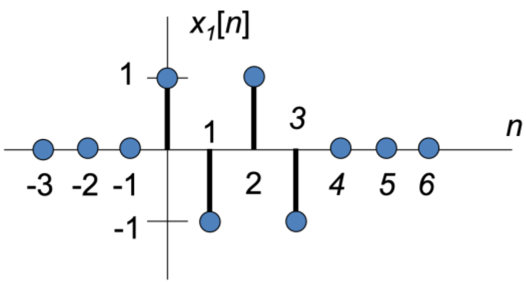


For each of the following input signals,
i. give a formula for output signal $y[n]$. *2 points each.*
ii. plot the output signal $y[n]$. *4 points each.*

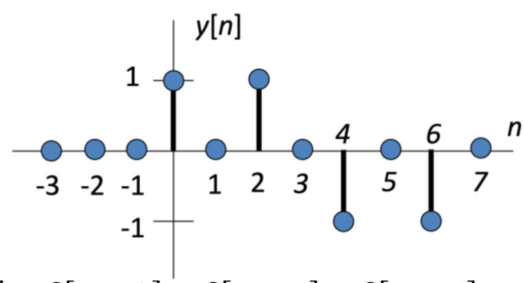
(a) $x_1[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$

Here, $x_1[n]$ has four non-zero values.

$y[n] = h[n] * x_1[n]$
Give a formula for $y[n]$
Plot $y[n]$




```
h = [1 1 1 1];
x1 = [1 -1 1 -1];
y = conv(h, x1);
% [1 0 1 0 -1 0 -1]
```

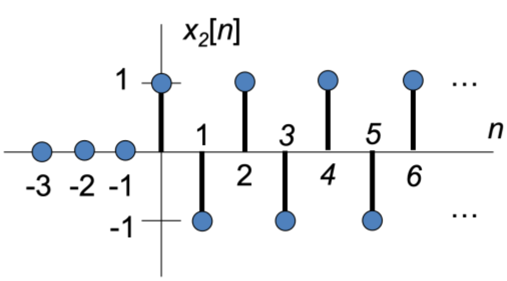


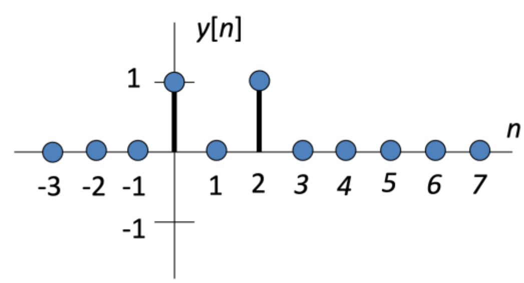
$\boldsymbol{y[n] = \delta[n] + \delta[n-2] - \delta[n-4] - \delta[n-6]}$

(b) $x_2[n] = (-1)^n\, u[n]$

Here, $x_2[n]$ is 0 for $n < 0$. For $n \geq 0$, $x_2[n]$ alternates between 1 and -1 indefinitely.

$y[n] = h[n] * x_2[n]$
Give a formula for $y[n]$
Plot $y[n]$



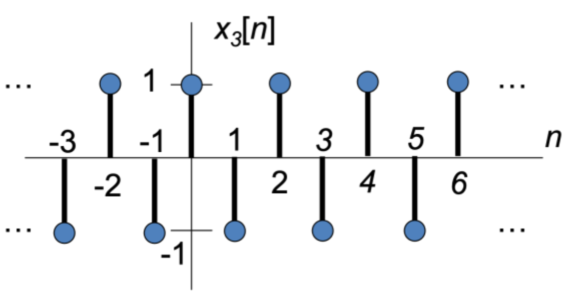


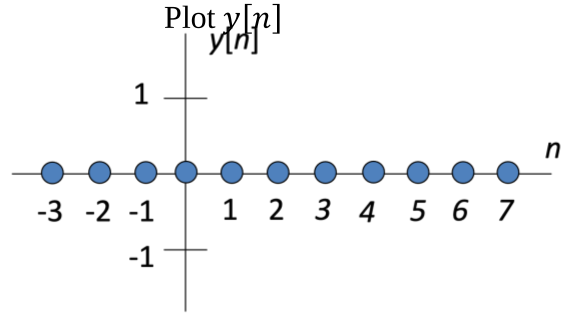
$\boldsymbol{y[n] = \delta[n] + \delta[n-2]}$

(c) $x_3[n] = (-1)^n$

Here, $x_3[n]$ alternates between 1 and -1 for all n.

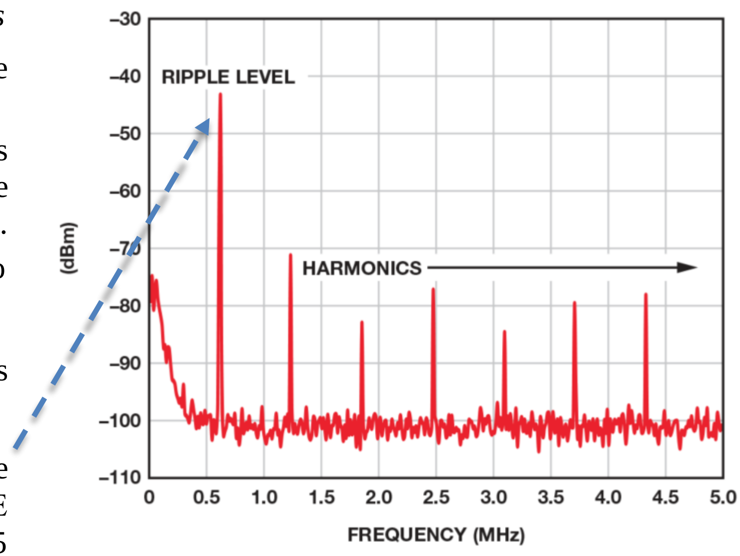
$y[n] = h[n] * x_3[n]$
Give a formula for $y[n]$
Plot $y[n]$





$\boldsymbol{y[n] = 0}$

Problem 3. Continuous-Time Filter Design. *16 points*



In power systems, a DC-to-DC converter changes the voltage level of a direct current (DC) signal.

A Buck converter provides high efficiency but produces undesirable output ripple and harmonics. The frequency plot on the right shows the first 7 harmonics.

Design a continuous-time linear time-invariant filter to

- pass frequencies between -0.4 MHz and 0.4 MHz
- eliminate the fundamental frequency f_0 and all its harmonics

(a) Estimate the fundamental frequency f_0 which is the frequency of the peak just below the text "RIPPLE LEVEL". The answer is somewhere between 0.5 and 0.75 MHz. Explain your reasoning. *4 points*.

From 0 to approximately 2.5 MHz, there are 4 harmonics, so $f_0 = \frac{2.5 \text{ MHz}}{4} = 6.25$ MHz.

Note: **Other answers for f_0 between 0.5 and 0.75 MHz were acceptable with justification.**

(b) What is the best description of the frequency selectivity of the continuous-time linear time-invariant filter— lowpass, highpass, bandpass, bandstop, allpass, or notch? Why? *4 points*.

Because we are passing frequencies below 0.4 MHz and eliminating an infinite number of harmonic frequencies at or above 0.5 MHz, this is a lowpass filter.

Note: **If we had been asked to pass frequencies below 0.4 MHz and eliminate only the seven harmonics shown above, then one could have said either a *bandstop filter* to attenuate frequencies from 0.5 MHz to 4.5 MHz and pass frequencies below 0.4 MHz and above 4.6 MHz or a *notch filter* that notches the seven harmonics and their negative frequency counterparts.**

(c) Give an equation for the impulse response of the linear-time invariant (LTI) filter and plot it in the continuous-time domain. *4 points*.

Approach #1: **Ideal filter. The impulse response would be a two-sided sinc pulse for all time whose frequency response would be an ideal lowpass filter (rectangular pulse) that would pass frequencies between -0.4 MHz to 0.4 MHz and eliminate all other frequencies.**

Approach #2: **Practical filter (averaging filter). Impulse response is a rectangular pulse with amplitude 1 for $-\frac{1}{2}T_0 < t < \frac{1}{2}T_0$ where the fundamental period $T_0 = \frac{1}{f_0}$. Frequency response would be a two-sided sinc function whose zero crossings would be at integer multiples of f_0.**

(d) Give an equation for the frequency response of the LTI filter and plot it in the continuous-time frequency domain from -5 MHz to 5 MHz. *4 points*.

See the answers to part (c) above.

The above plot is from Figure 2 in "Understanding Switching Regulator Output Artifacts Expedites Power Supply Design" by Aldrick Limjoco, Analog Devices.

Problem 4. Discrete-Time Filter Design. *18 points*.

People suffering from tinnitus, or ringing of the ears, hear a tone in their ears even when the environment is quiet. The tone is generally at a fixed frequency in Hz, denoted as f_c.

Filtering music to remove as much as possible of an octave of continuous-time frequencies from f_1 to f_2 that contains f_c as its center frequency can provide relief of tinnitus symptoms.

To cover an octave of frequencies, $f_2 = 2\, f_1$. With $f_c = ½\,(f_1 + f_2)$, we have $f_1 = (2/3)\, f_c$ and $f_2 = (4/3)\, f_c$.

This problem will ask you to design a sixth-order discrete-time linear time-invariant (LTI) infinite impulse response (IIR) filter to remove the octave of frequencies.

The **sampling rate** is f_s where $f_s > 4\, f_2$.

(a) What is the best description of the frequency selectivity of the continuous-time linear time-invariant filter— lowpass, highpass, bandpass, bandstop, allpass, or notch? Why? *3 points*.

Bandstop filter to attenuate frequencies from (2/3) f_c and (4/3) f_c where f_c is the tinnitus frequency in the audible range of 20 Hz and 20 kHz for a person with tinnitus.

(b) Give formulas for discrete-time frequencies $\hat{\omega}_1, \hat{\omega}_c,$ and $\hat{\omega}_2$ that correspond to continuous-time frequencies f_1, f_c and f_2, respectively. *3 points*.

$$\boldsymbol{\hat{\omega}_1 = 2\pi\frac{f_1}{f_s} \text{ and } \hat{\omega}_c = 2\pi\frac{f_c}{f_s} \text{ and } \hat{\omega}_2 = 2\pi\frac{f_2}{f_s}}$$

(c) Give formulas in terms of $\hat{\omega}_1, \hat{\omega}_c,$ and $\hat{\omega}_2$ for the pole and zero locations for the sixth-order discrete-time IIR filter which has 6 zeros and 6 poles. Every positive frequency has a negative frequency counterpart. So, if there is a zero at $z = e^{j\,\hat{\omega}_0}$, there's also a zero at $z = e^{-j\,\hat{\omega}_0}$. *9 points*.

Zeros are real-valued or occur in conjugate symmetric pairs. Same with the poles.

Filter should attenuate frequencies between $\hat{\omega}_1$ and $\hat{\omega}_2$ as well as between $-\hat{\omega}_2$ and $-\hat{\omega}_1$.

<u>Bandstop filter.</u> Zeros on or near the unit circle indicate the stopband.

Poles inside and near the unit circle indicate the passband(s).

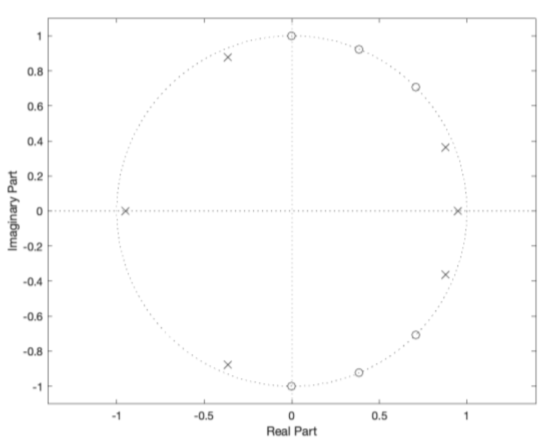
Zeros would be at frequencies $\hat{\omega}_1$, $\hat{\omega}_c$ and $\hat{\omega}_2$ as well as their negative values.

Because $f_s > 4\, f_2$, $\hat{\omega}_2$ will be between 0 and $\pi/2$.

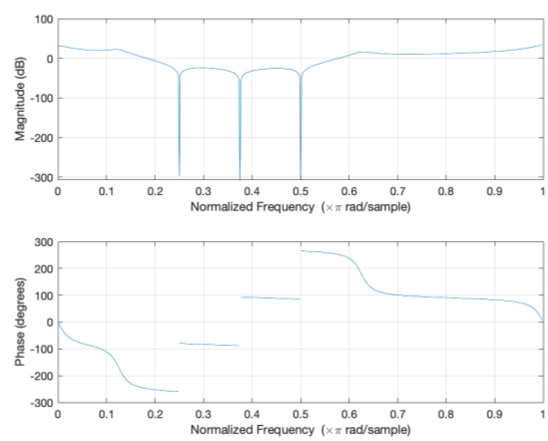
$$\textbf{Zeros}: e^{j\hat{\omega}_2}, e^{j\hat{\omega}_c}, e^{j\hat{\omega}_1}, e^{-j\hat{\omega}_1}, e^{-j\hat{\omega}_c}, e^{-j\hat{\omega}_2}$$

$$\textbf{Poles}: r\,e^{j\left(\frac{1}{2}\right)\hat{\omega}_1}, r, r\,e^{-j\left(\frac{1}{2}\right)\hat{\omega}_1}, re^{-j\left(\frac{5}{4}\right)\hat{\omega}_2}, -r \text{ and } r\,e^{j\left(\frac{5}{4}\right)\hat{\omega}_2} \text{ where } r = 0.95$$

(d) Draw the pole-zero diagram using the numeric values below. *3 points*.



f_1 = 2000 Hz
f_c = 3000 Hz
f_2 = 4000 Hz
f_s = 16000 Hz

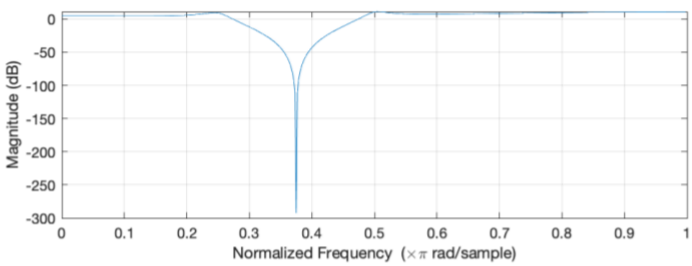


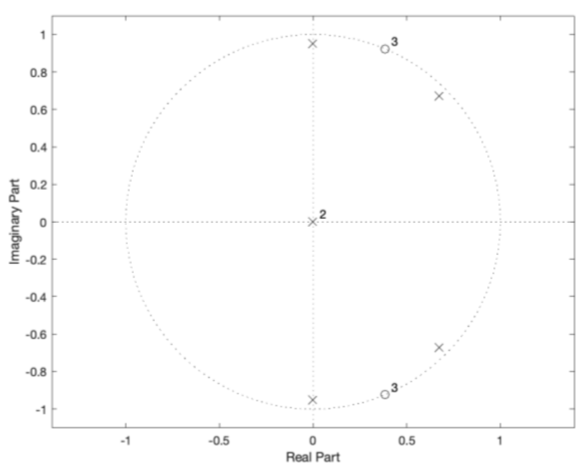
An alternate answer is on the next page.

```
% Writing MATLAB code is not required on the text
% This code is provide for additional insight into the answer in 2.4(d).
% Bandstop filter in prob. 2.4(d)
f1 =  2000;
fc =  3000;
f2 =  4000;
fs = 16000;

w1 = 2*pi*f1/fs;
wc = 2*pi*fc/fs;
w2 = 2*pi*f2/fs;

z1 = exp(j*w1);
zc = exp(j*wc);
z2 = exp(j*w2);
z1n = exp(-j*w1);
zcn = exp(-j*wc);
z2n = exp(-j*w2);
zeros = [z1 zc z2 z1n zcn z2n];
numer = poly(zeros);

r = 0.95;
p1 = r*exp(j*(1/2)*w1);
p2 = r;
p3 = r*exp(-j*(1/2)*w1);
p4 = r*exp(-j*(5/4)*w2);
p5 = -r;
p6 = r*exp(j*(5/4)*w2);
poles = [p1 p2 p3 p4 p5 p6];
denom = poly(poles);

figure;
[hz1, hp1, ht1] = zplane(numer, denom);
set(findobj(hz1, 'Type', 'line'), 'Color', 'k');
set(findobj(hp1, 'Type', 'line'), 'Color', 'k');
set(findobj(ht1, 'Type', 'line'), 'Color', 'k');

figure;
freqz(numer, denom);
```

Student answer for parts (c) and (d)

(c) Place 3 zeros on unit circle at angle $\hat{\omega}_c$ and same goes at angle $\widehat{-\omega}_c$. Place 4 poles at radius 0.95 and angles $\hat{\omega}_1, -\hat{\omega}_1, \hat{\omega}_2, -\hat{\omega}_2$. Place 2 poles at $z = 0$.





Problem 5. Continuous-Time Sinusoidal Amplitude Modulation. *16 points*.

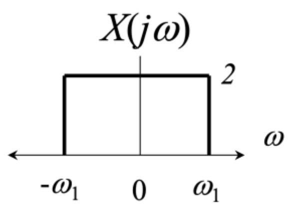
Continuous-time sinusoidal amplitude modulation multiplies the input signal $x(t)$ by a sinusoidal signal of fixed frequency ω_c in rad/s to give the output signal $y(t)$ where

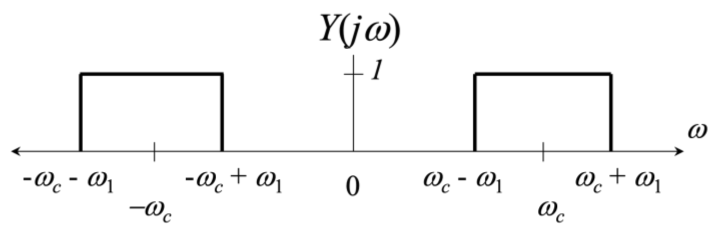
$$y(t) = x(t)\ \cos(\omega_c\, t)$$

By taking the Fourier transform of both sides, we obtain the Modulation Property:

$$Y(j\,\omega) = \frac{1}{2}\, X(j\,(\omega + \omega_c)) + \frac{1}{2}\, X(j\,(\omega - \omega_c))$$

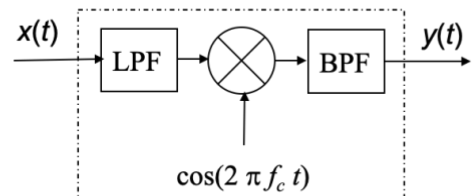
The term $\frac{1}{2}\, X(j\,(\omega + \omega_c))$ shifts the frequency content of $X(j\,\omega)$ left in frequency by ω_c and scales the amplitude by ½ and the term $\frac{1}{2}\, X(j\,(\omega - \omega_c))$ shifts the frequency content of $X(j\,\omega)$ right in frequency by ω_c and scales the amplitude by ½. Here's an example using an ideal lowpass spectrum for $X(j\,\omega)$:



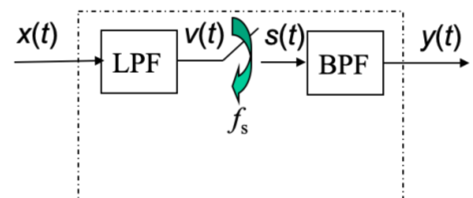


Please use the above Fourier transforms for $x(t)$ and $y(t)$ throughout this problem.

Mixer #1. In practice, we apply a lowpass filter (LPF) to enforce the lowpass bandwidth of $X(j\,\omega)$ to be ω_1 and a bandpass filter (BPF) to enforce the bandpass bandwidth of $Y(j\,\omega)$ to be $2\omega_1$ centered at ω_c, as plotted above. Note that $\omega_c = 2\,\pi\, f_c$.



Mixer #2. Mixer #1 can be simplified by replacing the analog multiplier and cosine generator with a sampling block operating at sampling rate f_s. We keep the LPF and BPF filters the same.



Assume all the LPF and BPF filters are ideal filters.

(a) Plot $V(j\,\omega)$. *4 points.*

(b) Plot $S(j\,\omega)$. *6 points.*

(c) Give formulas that describe all the possible values for the sampling rate f_s so that the **mixer #2** implements sinusoidal amplitude modulation. *6 points.*

Problem 6. Discrete-Time Mystery Systems. *14 points*.

You're trying to identify unknown discrete-time systems.

You input a discrete-time chirp signal $x[n]$ and look at the output to figure out what the system is.

The discrete-time chirp is formed by sampling a chirp signal that sweeps 0 to 8000 Hz over 0 to 5s

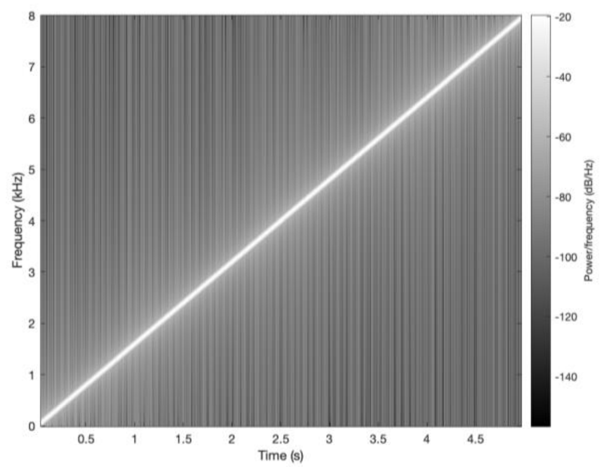
$$x(t) = \cos(2\pi f_1 t + 2\pi \mu t^2)$$

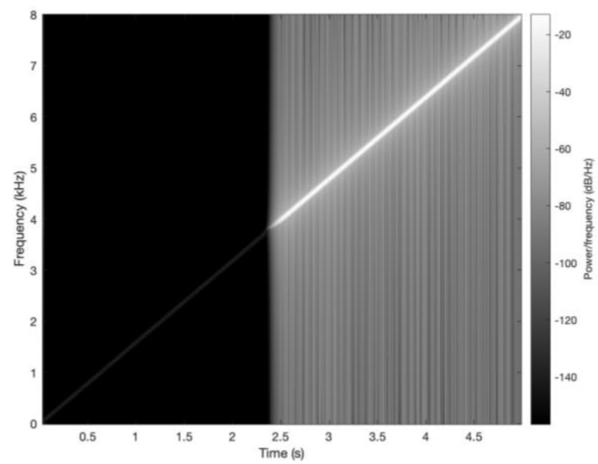
where $f_1 = 0$ Hz, $f_2 = 8000$ Hz, and $\mu = \frac{f_2 - f_1}{2\, t_{\max}} = \frac{8000 \text{ Hz}}{10\, s} = 800 \text{ Hz}^2$. Sampling rate f_s is 16000 Hz.

In part (a) and (b) below, identify the unknown system as one of the following **with justification**:

1. filter – give selectivity (lowpass, highpass, bandpass, bandstop) and passband/stopband frequencies
2. pointwise nonlinearity – give the integer exponent k to produce the output $y[n] = x^k[n]$

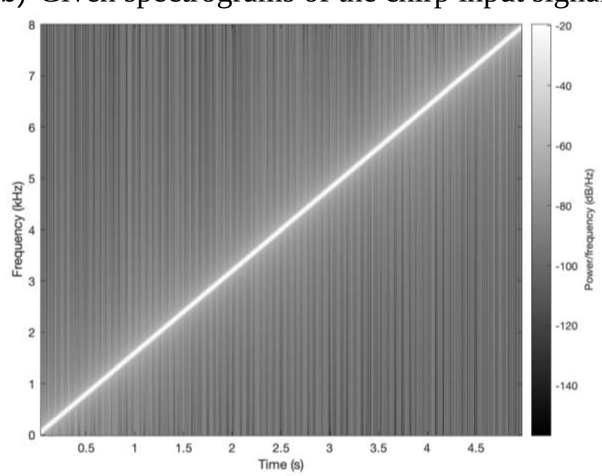
(a) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *7 points*.

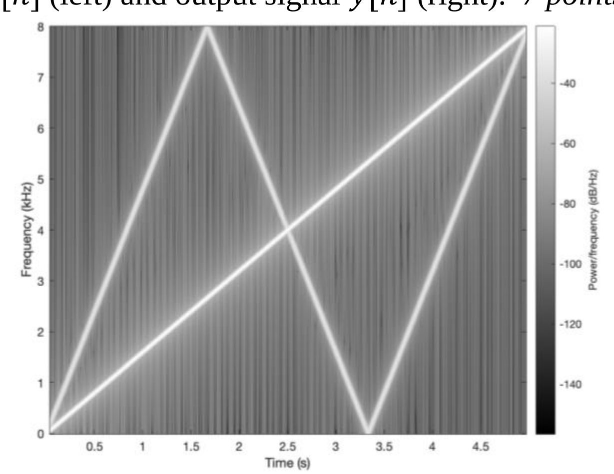




In the output spectrogram, principal frequencies of the chirp input signal between 0 and 4 kHz are severely attenuated and higher principal frequencies are passed. No new frequencies are created, so it is likely an LTI filter. Highpass filter. **See the next page for the Matlab code.**

(b) Given spectrograms of the chirp input signal $x[n]$ (left) and output signal $y[n]$ (right). *7 points.*





In the output spectrogram, new frequencies are being created, so it's not an LTI system. At any point in time from 0s to 1.67s, the frequency component in the output spectrogram that rises

from 0 Hz to 8000 Hz is three times the principal frequency component in the input spectrogram; input frequencies that exceed 2667 Hz will be tripled and then alias, which makes the shape of an italicized letter N. The output spectrogram also has a copy of the input signal. This is a cubing block, i.e., a pointwise nonlinearity with $k = 3$. When inputting $\cos(\omega_0\ t)$ into a cubing block, the output is $\cos^3(\omega_0\ t) = \frac{3}{4}\cos(\omega_0\ t) + \frac{1}{4}\cos(3\ \omega_0\ t)$ per slide 3-9.

```
%% Midterm Problem 2.6
fs = 16000;
Ts = 1 / fs;
tmax = 5;
t = 0 : Ts : tmax;

%% Create chirp signal
f1 = 0;
f2 = fs/2;
mu = (f2 - f1) / (2*tmax);
x = cos(2*pi*f1*t + 2*pi*mu*(t.^2));

%% (a) highpass filter
fnyquist = fs/2;
fstop = 3800;
fpass = 4200;
ctfrequencies = [0 fstop fpass fnyquist];
idealAmplitudes = [0 0 1 1];
pmfrequencies = ctfrequencies / fnyquist;
filterOrder = 300;
h = firpm( filterOrder, pmfrequencies, idealAmplitudes );
h = h / sum(h .^ 2);

y = conv(x, h);

%%% Spectrogram parameters
blockSize = 1024;
overlap = 1023;

%%% Plot spectrogram of input signal
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(y, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%% (b) cubic block
figure;
spectrogram(x, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;

%%% Plot spectrogram of output signal
figure;
spectrogram(x.^3, blockSize, overlap, blockSize, fs, 'yaxis');
colormap gray;
```